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Statistics of quantum transfer of noninteracting fermions in multiterminal junctions

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Similarly to the recently obtained result for two-terminal systems, we show that there are constraints on the full counting statistics for noninteracting fermions in multiterminal contacts. In contrast to the two-terminal result, however, there is no factorization property in the multiterminal case.

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I. INTRODUCTION

The problem of full counting statistics (FCS) of electronic charge transfer has been addressed since long time[1] and the particular model of noninteracting fermions has been studied in detail in various setups. The FCS for transfer of noninteracting fermions is given by the Levitov-Lesovik determinant formula[1–5] valid at arbitrary temperature and for an arbitrary time evolution of the scatterer. Recently, some properties of this result have been elucidated. First, in the particular case of charge transfer driven by a time-dependent bias voltage at zero temperature, the resulting FCS enjoys certain symmetries.[6–8] Second, in the more general case of an arbitrary time-dependent scatterer and at arbitrary temperature, it has been shown that the FCS is factorizable into independent single-particle events.[8,9]

In the present work, we generalize the result of Refs. 8 and 9 to a multiterminal setup. As in those works, we address the problem of determining which multichannel charge transfers are possible and which are not in an arbitrary quantum pump, in the model of noninteracting fermions. In the two-terminal case, the constraint derived in Refs. 8 and 9 is exact. In the multiterminal case, however, the problem is more complicated, and we have only partially solved it: we have formulated a *necessary* constraint (a "convexity condition") on the charge-transfer statistics, without a proof (or a counterexample) that this constraint is sufficient. Also, there is no obvious physical interpretation of this constraint: we show that, unlike in the two-terminal case, our constraint cannot be interpreted as a factorization property of the charge-transfer statistics. This work is partly based on the results reported in Ref. 10.

II. DETERMINANT FORMULA

We first present notation and review the Levitov–Lesovik determinant formula[1–5] for charge transfer of noninteracting fermions in application to a multilead setup. The notation and argument is fully parallel to that in Ref. 9 where the two-lead case was considered.

We consider a contact with L leads, connected by an arbitrary time-dependent scatterer (see Fig. 1). To each lead (numbered $i=1,\ldots,L$) we associate a "counting field"[1] λ_i and a projector operator P_i acting in the single-particle Hilbert space. The leads are defined in such a way that

$$\sum_{i=1}^{L} P_i = 1. \tag{1}$$

Then the probabilities of the multilead charge transfers can be determined from the generating function

$$\chi(\lambda_1,\ldots,\lambda_L) = \mathrm{Tr}(\hat{\rho}_0 \hat{U}^\dagger e^{i\boldsymbol{\lambda}\hat{\mathbf{P}}} \hat{U} e^{-i\boldsymbol{\lambda}\hat{\mathbf{P}}})/\mathrm{Tr}\,\hat{\rho}_0. \tag{2}$$

Here the trace is taken in the multiparticle Fock space, $\hat{\rho}_0$ is the initial density matrix, $\hat{U}$ is the multiparticle evolution operator. We also use the shorthand notation $\boldsymbol{\lambda}\hat{\mathbf{P}}=\Sigma_i \lambda_i \hat{P}_i$, where $\hat{P}_i$ is the multiparticle operator (a fermionic bilinear)[9] constructed from the projector P_i (it counts the particles in the lead i). As in the two-lead problem,[9] under the assumption that $\hat{\rho}_0$ commutes with $\hat{P}_i$ (the absence of entanglement in the initial state), the Fourier components of the generating function (2) give the charge-transfer probabilities $P_{q_1,\ldots,q_L}$,

$$\chi(\lambda_1,\ldots,\lambda_L) = \sum_{q_1,\ldots,q_L=-\infty}^{\infty} P_{q_1,\ldots,q_L} \exp\left(i\sum_{i=1}^{L}\lambda_i q_i\right). \tag{3}$$

Those probabilities are only nonzero for charge-conserving transfers with $\Sigma_i q_i=0$. This charge conservation corresponds to the symmetry of the generating function with respect to a simultaneous shift of all variables,

$$\chi(\lambda_1,\ldots,\lambda_L) = \chi(\lambda_1+\delta\lambda,\ldots,\lambda_L+\delta\lambda). \tag{4}$$

As in the two-lead case, we define the complex variables

$$u_i = e^{i\lambda_i}, \quad i=1,\ldots,L, \tag{5}$$

and consider the generating function as a function of u_i.

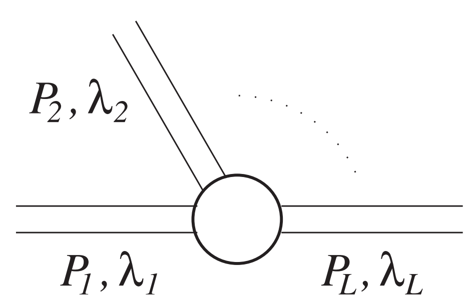

FIG. 1. A schematic figure of the multiterminal contact. To each of the L leads, there corresponds a single-particle projector P_i and a counting variable λ_i.

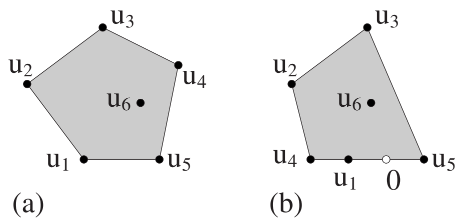

FIG. 2. (a) Illustration of the definition of the convex envelope (convex hull). The shaded region shows the convex envelope of the points $u_1,\ldots,u_6$ in the complex plane. If the points $u_1,\ldots,u_6$ correspond to a root of the generating function, then condition 1 of the constraint claims that zero must belong to the shaded region. (b) Illustration of condition 2 of the constraint. In this figure (with the points $u_1,\ldots,u_6$ corresponding to a root of the generating function), the points u_2, u_3, and u_6 can be changed arbitrarily, and the set of points will still give a root of the generating function.

As in Ref. 9, we assume, in addition to the absence of entanglement of the initial state, that both $\hat{\rho}_0$ and $\hat{U}$ are exponentials of fermionic bilinears (which reflects our assumption of noninteracting fermions). Under those assumptions, we repeat the calculation of Ref. 9 and arrive at the resulting determinant formula

$$\chi(\lambda_1,\ldots,\lambda_L)=\det[1+n_F(U^\dagger e^{i\lambda\mathbf{P}}Ue^{-i\lambda\mathbf{P}}-1)], \quad (6)$$

which involves only operators in the single-particle Hilbert space with the occupation-number operator

$$n_F=\frac{\rho_0}{\rho_0+1}. \quad (7)$$

III. CONVEXITY CONDITION

Similarly to the trick employed in Ref. 9, we can rewrite the determinant formula by defining the hermitian "effective-transparency operators"

$$\tilde{X}_{(i)}=(1-n_F)P_i+n_F^{1/2}U^\dagger P_i U n_F^{1/2}. \quad (8)$$

After simple algebra [using the completeness relation (1)], one can re-express the generating function (6) as

$$\chi(u_1,\ldots,u_L)=\det\left[e^{-i\lambda\mathbf{P}}\sum_{i=1}^{L}u_i\tilde{X}_{(i)}\right]. \quad (9)$$

The eigenvalues of the operators $\tilde{X}_{(i)}$ are bounded between 0 and 1, which allows us to prove a certain constraint on the zeroes (roots) of the generating function (9). An elegant form of this constraint can be formulated in terms of the *convex envelope* (convex hull) $H_c(X)$ of a given set of complex numbers X: a minimal convex set containing X [see Fig. 2(a)]. The constraint may now be cast in the form of two conditions that need to be satisfied:

(1) For any root of the characteristic function $\chi(u_1,\ldots,u_L)=0$, the convex envelope $H_c(\{u_1,\ldots,u_L\})$ contains zero.

(2) If $\chi(u_1,\ldots,u_L)=0$ and if zero belongs to the *boundary* of $H_c(\{u_1,\ldots,u_L\})$, then those of the points $\{u_1,\ldots,u_L\}$ that do not lie on the straight segment of the boundary of $H_c(\{u_1,\ldots,u_L\})$ containing zero, can be arbitrarily changed while still satisfying the equation $\chi(u_1,\ldots,u_L)=0$ [Fig. 2(b)].

The proof of condition 1 is easy: if $|\Psi\rangle$ is a zero mode of the operator in the determinant (9), then

$$\sum_{i=1}^{L}u_i\langle\Psi|\tilde{X}_{(i)}|\Psi\rangle=0. \quad (10)$$

Since all the coefficients $\langle\Psi|\tilde{X}_{(i)}|\Psi\rangle$ are non-negative real numbers (whose sum equals one), zero belongs to the convex envelope of $u_1,\ldots,u_L$.

To prove condition 2, consider again a root $(u_1,\ldots,u_L)$ of the generating function and the corresponding zero mode $|\Psi\rangle$. If zero lies at the boundary of the convex envelope $H_c(\{u_1,\ldots,u_L\})$, then the linear combination (10) contains nonvanishing coefficients $\langle\Psi|\tilde{X}_{(i)}|\Psi\rangle$ *only* for variables u_i which belong to the same straight segment of the boundary containing zero. All the other coefficients necessarily vanish, which, by virtue of the non-negativity of $\tilde{X}_{(i)}$, implies $\tilde{X}_{(i)}|\Psi\rangle=0$. Therefore all those variables u_i may be changed arbitrarily while $|\Psi\rangle$ will remain a zero mode. This completes the proof of condition 2.

We can make several comments on the obtained result. First, in the particular case of two leads ($L=2$), this constraint is equivalent to that found in Ref. 9 (the variable u in that work corresponds to the ratio u_1/u_2 in our present notation). Second, while our constraint is a necessary condition for realizability of a given statistics in a noninteracting fermionic system, we could not determine if it is also a sufficient one. Moreover, we do not have any algorithm which would determine if a given charge-transfer statistics is realizable (or design a suitable quantum evolution if it is). Those interesting questions are left for future studies. Third, our criterion is technically difficult to check in its full formulation for all roots $(u_1,\ldots,u_L)$. However, for practical applications, one may test the constraint on suitably chosen families of roots (e.g., one-parametric families),[13] either analytically or numerically.

IV. NONFACTORIZABILITY

In the two-terminal case, the "convexity condition" derived above implies a factorizability of the charge-transfer statistics: the probabilities of a given charge transfer are the same as in a superposition of some single-electron transfer processes (whose transfer probabilities depend in a nontrivial way on the evolution of the quantum system). One can see that it is not the case in the multiterminal ($L>2$) case.

This can be most easily demonstrated with a counterexample involving only a finite number of electrons (in the wave packet formalism of Ref. 11, to which our result is also applicable). Consider two fermions sent into a stationary multiterminal contact along two terminals (labeled 1 and 2) with exactly the same shape of wave packets (Fig. 3). Then, due to the Fermi statistics of particles, the probabilities to have both fermions scattered to the same lead vanish. The

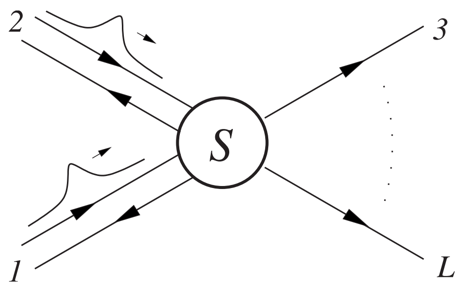

FIG. 3. A counterexample proving nonfactorizability of the full counting statistics for multiterminal contacts. Two fermions are sent to a time- and energy-independent L lead scatterer (with $L \geq 3$) along the leads 1 and 2 in the shape of exactly identical wave packets, synchronized in time.

resulting generating function will therefore have the form

$$\chi(u_1, \ldots, u_L) = \frac{1}{u_1 u_2} \sum_{i<j} \alpha_{ij} u_i u_j, \tag{11}$$

where $\alpha_{ij} = |s_{1i}s_{2j} - s_{2i}s_{1j}|^2$ are the probabilities of various two-particle transfer events constructed out of the single-particle scattering amplitudes s_{ij} (which are assumed to be time and energy independent). On the other hand, the factorizability of the charge transfer would imply

$$\chi(u_1, \ldots, u_L) = \frac{1}{u_1 u_2}\left(\sum_i p_i u_i\right)\left(\sum_i p_i' u_i\right), \tag{12}$$

for some probabilities p_i and p_i'. One can verify that if one considers a statistics (11) with all α_{ij} nonzero (which is possible), then such a statistics is not factorizable in form (12).

V. CONCLUSION

To summarize, we have considered the problem of possible full counting statistics for noninteracting fermions in coherent multiterminal systems. We have obtained a necessary condition for a full counting statistics to be realizable. Like in the two-terminal case,[9] this condition may be used to prove impossibility of certain sets of charge-transfer probabilities (one can easily construct examples of such impossible statistics).

At the same time, the problem of designing an actual "quantum pump" for a given charge-transfer statistics (or even merely proving its *possibility*) appears much more difficult in the multiterminal case than in the two-terminal one. While in the two-terminal case, the full counting statistics of noninteracting fermions is conveniently parameterized by the spectral density of "effective transparencies,"[9] we are not aware of a similar parameterization in the multiterminal case. In the formulation with a finite number of particles,[11] even the question of the dimensionality of the space of all possible full counting statistics remains open. All those interesting questions deserve further study, in particular, in the context of using quantum contacts for generating entangled states.[12]

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[13] A good practical test of the constraint is obtained by restricting u_i to a one-parameter family $u_i = a_i z + b_i$ for some fixed sets of real numbers a_i and b_i with the condition that all $a_i \geq 0$ (nothing is assumed about b_i). Then the equation $\chi(z) = 0$ must either have only real roots z or be identically satisfied for all z. This form of the constraint is convenient for numerical tests by using a large number of randomly chosen vectors a_i and b_i.